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Addendum

Addendum (Helmholtz decomposition) to: "Potential flow of viscous fluids: Historical notes" [Int. J. Multiphase Flow 32 (2006) 285–310]

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A framework for embedding the study of potential flow of viscous fluids, in which no special flow assumptions are made, is provided by Hemholtz decomposition, which says that every vector field $\mathbf{u}$ can be decomposed into a rotational part $\mathbf{v}$ and an irrotational part $\nabla\phi$

$$\mathbf{u} = \mathbf{v} + \nabla\phi,$$

where

$$\operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{v} + \nabla^2\phi,$$
$$\operatorname{curl}\mathbf{u} = \operatorname{curl}\mathbf{v}.$$

If the field is solenoidal then

$$\operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{v} = 0 \quad \text{and} \quad \nabla^2\phi = 0.$$

The stress in a Newtonian incompressible fluid is given by

$$\mathbf{T} = -p\mathbf{1} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) = \mathbf{T} = -p\mathbf{1} + 2\mu\mathbf{D}[\mathbf{u}] = -p\mathbf{1} + \mu\left(\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}}\right) + 2\mu\nabla\otimes\nabla\phi.$$

Every flow has an irrotational viscous stress. It does not act internally because it is self equilibrated. It acts at boundaries where it is not equilibrated. The viscous dissipation is given by

$$\Phi = \int_V 2\mu D_{ij}D_{ij}\mathrm{d}V = \int 2\mu D_{ij}[\mathbf{v}]D_{ij}[\mathbf{v}]\mathrm{d}V + 4\mu\int D_{ij}[\mathbf{v}]\frac{\partial^2\phi}{\partial x_i\,\partial x_j}\mathrm{d}V + 2\mu\int\frac{\partial^2\phi}{\partial x_i\,\partial x_j}\frac{\partial^2\phi}{\partial x_i\,\partial x_j}\mathrm{d}V.$$

Every flow has an irrotational viscous dissipation, In regions V' of irrotational flow $\mathbf{v} = 0$ (outside boundary layers $\mathbf{v} \neq 0$)

$$\mathbf{T} = -p\mathbf{1} + 2\mu\nabla\otimes\nabla\phi,$$
$$\Phi = 2\mu\int_{V'}\frac{\partial^2\phi}{\partial x_i\,\partial x_j}\frac{\partial^2\phi}{\partial x_i\,\partial x_j}\mathrm{d}V.$$

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No one in his right mind would put $\mu = 0$.

The decomposition of the velocity into rotational and irrotational parts holds at each and every point and varies from point to point in the flow domain. Various possibilities for the balance of these parts at a fixed point and the distribution of these balances from point to point can be considered.

(i) The flow is purely irrotational or purely rotational. These two possibilities do occur but are not typical.
(ii) Typically the flow is mixed with rotational and irrotational components at each point. For example, in Lamb's exact solution for the decay of free gravity waves due to viscosity, the solution is given by a potential ϕ and a stream function ψ:

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}, \quad \frac{p}{\rho} = -\frac{\partial \phi}{\partial t} - gy,$$

where $\nabla^2 \phi = 0$ and $\partial \psi / \partial t = v \nabla^2 \psi$. The stream function gives rise to the rotational part of the flow. This example is typical; many problems are solved this way. Even in problems of Stokes flow past bodies the velocity has an irrotational component.
(iii) Purely irrotational flow calculations are useful in cases in which the flow is segregated; in particular, in cases in which the rotational flow is confined to boundary layers.
(iv) The flow near stagnation points is irrotational. Stagnation points can occur in interior flows and at boundaries. Stagnation points enter strongly into the motion of particulates, into the turning couples which determine the flow orientation of long bodies, the microstructures of flowing spherical bodies and the shape of moving gas bubbles. Interior stagnation points are also important and the role they play in the organization of flows may be underappreciated.